A Frobenius manifold for ℓ–Kronecker quiver

Takumi Otani

Osaka university

April 6, 2021

joint work with Akishi Ikeda, Yuuki Shiraishi and Atsushi Takahashi.

Recall 3 different constructions of Frobenius manifolds:

- (GW) Gromov–Witten theory.
- (Def.) Deformation theory.
- (Weyl) Invariant theory of a Weyl group.

The construction (Weyl) is related with (Def.) by the period mapping.

Example (ADE)

(Def.)	$\longleftrightarrow$	(Weyl)
ADE singularity		ADE root system

The isomorphism of Frobenius manifold between (Def.) and (Weyl) is induced by the period mapping of the primitive form $\zeta = dz$.

The construction (Weyl) is known a few cases;

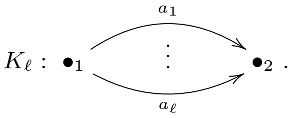
- finite Weyl group [Saito, Saito–Yano–Sekiguchi, Dubrovin],
- extended affine Weyl group [Dubrovin–Zhang, Dubrovin–Zhang–Zuo, Zuo],
- elliptic Weyl group [Saito, Satake, Dubrovin, Bertola],

⋮

Problem

Establish a construction of Frobenius structures by the invariant theory of a Weyl group for a given root system.

Let K_ℓ be the ℓ–Kronecker quiver:



It is known that an acyclic finite quiver induces a root system of a Kac–Moody Lie algebra. In the case of the ℓ–Kronecker quiver K_ℓ

- if $\ell = 1$, $K_1 = A_2$ quiver is of finite type,
- if $\ell = 2$, $K_2 = \widetilde{A_1}$ quiver is of affine type,
- if $\ell \geq 3$, K_ℓ is of indefinite type.

The Kac–Moody Lie algebra associated with the ℓ-Kronecker quiver K_ℓ with $\ell \geq 3$ is one of the most basic class of indefinite types.

Background : Bridgeland stability condition

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $K_0(\mathcal{D})$ the Grothendieck group of $\mathcal{D}$.

A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ consists of

- $Z : K_0(\mathcal{D}) \to \mathbb{C}$; group homomorphism (called a central charge),
- $\mathcal{P}(\phi)$: additive full sub categories ($\phi \in \mathbb{R}$),

satisfying some axioms (e.g. Harder–Narasimhan property).

It is shown by Bridgeland that the space of all stability conditions $\mathrm{Stab}(\mathcal{D})$ has the structure of a complex manifold.

Conjecture 1 (Takahashi).

$f : \mathbb{C}^3 \to \mathbb{C}$: ADE singularity,
$F : \mathbb{C}^3 \times S \to \mathbb{C}$: universal unfolding of f ($S \cong \mathbb{C}^n$),
$\vec{\Delta}$: Dynkin quiver corresponding to f.
Then there should exist a biholomorphic map

$$\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta})) \cong S.$$

In particular, $\mathrm{Stab}(\mathcal{D}^b(\vec{\Delta}))$ has a Frobenius structure induced by the Frobenius manifold S constructed by the deformation theory and primitive forms.

Bridgeland–Qiu–Surtherland proved this conjecture in the case of A_2 type.
The case of A_n type was proved by Haiden–Katzarkov–Kontsevich.

Conjecture 1 can be generalized to the affine type.
Haiden–Katzarkov–Kontsevich also showed the case of $\widetilde{A_{p,q}}$ type.

Intersection form of a Frobenius manifold

Let $(M, \eta, \circ, e, E)$ be a Frobenius manifold of rank n and dimension d.

Let $\nabla : \mathcal{T}_M \to \mathrm{End}_{\mathcal{O}_M}(\mathcal{T}_M)$ be the Levi-Civita connection with respect to η. Recall that there exists a flat coordinate system $(t^1, \cdots, t^n)$ and the Frobenius potential $\mathcal{F} \in \mathcal{O}_M$. That is, we have

- $e = \partial_1, \ \ \mathrm{Ker}\nabla \cong \bigoplus_{i=1}^n \mathbb{C}_M \cdot \partial_i$
- η naturally gives a $\mathbb{C}_M$-bilinear $\eta : \mathrm{Ker}\nabla \times \mathrm{Ker}\nabla \to \mathbb{C}_M$,
- $E = \sum_{i=1}^n \left[(1-q_i)t^i + c_i\right] \partial_i$, if $q_i \neq 1$ then $c_i = 0$,
- $(C_{ijk} =) \ \eta(\partial_i \circ \partial_j, \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}$,
- $E\mathcal{F} = (3-d)\mathcal{F} +$ (quadratic terms in $t^2, \cdots, t^n$),

where $\partial_i = \dfrac{\partial}{\partial t^i}$.

We introduce an invariant of Frobenius manifolds. It plays an important role in the Weyl group invariant theory.

Definition 2.

Define a symmetric $\mathcal{O}_M$-bilinear form $g : \mathcal{T}_M \times \mathcal{T}_M \to \mathcal{O}_M$ by

$$g(\delta, \delta') := \eta(E^{-1} \circ \delta, \delta').$$

It induces a symmetric $\mathcal{O}_M$-bilinear form on Ω^1_M. We call this symmetric $\mathcal{O}_M$-bilinear form $g : \Omega^1_M \times \Omega^1_M \to \mathcal{O}_M$ the intersection form of the Frobenius manifold.

On the flat coordinate system $(t^1, \cdots, t^n)$, the intersection form g is given by

$$g(dt^i, dt^j) = \sum_{a,b=1}^{n} \eta^{ia}\eta^{jb} E\partial_a\partial_b\mathcal{F}, \quad \eta^{ia} := \eta(dt^i, dt^a).$$

Let $(M, \eta, \circ, e, E)$ be a Frobenius manifold of rank 2 and dimension d. In this case, Frobenius potentials is classified.

Proposition 3 (Dubrovin).

Let (t^1, t^2) *be a flat coordinate of the Frobenius manifold* $(M, \eta, \circ, e, E)$. *If* $d \neq \pm 1, 3$, *then the Frobenius potential* $\mathcal{F}$ *are given by*

$$\mathcal{F}(t^1, t^2) = \frac{\eta_{12}}{2}(t^1)^2 t^2 + c(t^2)^{\frac{3-d}{1-d}},$$

where $\eta_{12} \in \mathbb{C} \backslash \{0\}, \ c \in \mathbb{C}$.

Quiver and root system

Let $Q = \{Q_0, Q_1\}$ be a connected finite acyclic quiver, and set $Q_0 = \{1, \ldots, n\}$.

A matrix $A_Q = (a_{ij})$ of size n defined by

$$a_{ij} := 2\delta_{ij} - (q_{ij} + q_{ji}), \quad q_{ij} := \#\{i \to j \in Q_1\}, \quad \text{for } i, j \in Q_0,$$

is called the generalized Cartan matrix of Q.

ℓ-Kronecker quiver

For $Q = K_\ell$, the generalized Cartan matrix is given by

$$A_{K_\ell} = \begin{pmatrix} 2 & -\ell \\ -\ell & 2 \end{pmatrix}.$$

- If $\ell = 1$, A_{K_1} is positive definite matrix (finite type),
- If $\ell = 2$, A_{K_2} is semi-positive (affine type),
- If $\ell \geq 3$, A_{K_ℓ} is indefinite (indefinite type).

Let us consider a root system associated with the quiver Q:

- Define a free abelian group L by

$$L := \bigoplus_{i=1}^{n} \mathbb{Z} \cdot \alpha_i.$$

  Here, α_i is a formal generator and called the simple root of $i \in Q_0$.

- Define $\mathbb{Z}$-bilinear form $I : L \times L \to \mathbb{Z}$ by $I(\alpha_i, \alpha_j) := a_{ij}$.
- For each $i \in Q_0$, define a reflection $r_i \in \mathrm{Aut}(L, I)$ by

$$r_i(\lambda) := \lambda - I(\lambda, \alpha_i)\alpha_i, \quad \lambda \in L.$$

The Weyl group W associated with the quiver Q is a group generated by reflections:

$$W := \langle r_1, \dots, r_n \rangle \subset \mathrm{Aut}(L, I).$$

In particular, there is a "special" element c given by $c := r_1 r_2 \cdots r_n \in W$. This element $c \in W$ is called a Coxeter transformation.

Define the set of real roots Δ^{re} by

$$\Delta^{\mathrm{re}} := \{ w(\alpha_i) \in L \mid w \in W,\ i \in Q_0 \}.$$

Root system via derived category

In the categorical point of view, the root system associated with Q is given as follows:

Let $\mathcal{D} := \mathcal{D}^b(Q)$ be the derived category of finitely generated $\mathbb{C}Q$-modules and S_i the simple module corresponding to $i \in Q_0$.
Consider

- The Grothendieck group $K_0(\mathcal{D}) \cong \bigoplus_{i \in Q_0} \mathbb{Z}[S_i]$,
- The symmetrized Euler form $I_{\mathcal{D}} := \chi + \chi^T : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \to \mathbb{Z}$,

Then we have $(K_0(\mathcal{D}), I_{\mathcal{D}}) \cong (L, I)$. Moreover,

- The coxeter transformation is $c = -[\mathcal{S}_Q] = -\chi^{-1}\chi^T$, where $\mathcal{S}_Q$ is the Serre functor of $\mathcal{D}$.

Let $Q = \vec{\Delta}$ be a Dynkin quiver.

The Weyl group acts on the Cartan subspace $\mathfrak{h} := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) \cong \mathbb{C}^n$. In the case, the Coxeter transformation c has finite order. Hence, define $h \in \mathbb{Z}_{\geq 1}$ by the order of c and call it the Coxeter number.

Theorem 4 (Saito, Saito–Yano–Sekiguchi, Dubrovin).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ *of rank* n *and dimension* $d = 1 - \dfrac{2}{h}$ *on* $\mathfrak{h}/W$ *satisfying*

1. *The intersection form* g *coincides with the Cartan matrix* $A_{\vec{\Delta}}$.
2. *There exist* W*-invariant homogeneous polynomials* $t^1, \cdots, t^n$ *such that* $(t^1, \ldots, t^n)$ *is a (global) flat coordinate system of the Frobenius manifold.*
3. *The Euler vector field* E *is given by*

$$E = \sum_{i=1}^{n} \frac{\deg t^i}{h} t^i \frac{\partial}{\partial t^i}.$$

The Frobenius structure in Theorem 4 is based on Chevalley's Theorem;

Theorem 5 (Chevalley's Theorem).

Let $\alpha_i^* \in \mathfrak{h}$ be the dual (fundamental co-weight) of $\alpha_i \in \mathfrak{h}^*(:= L \otimes_{\mathbb{Z}} \mathbb{C})$ and $(x^1, \cdots, x^n)$ the linear coordinate with respect to $\{\alpha_1^*, \cdots, \alpha_n^*\}$. We have

1. *The W-invariant subring $\mathbb{C}[\mathfrak{h}]^W$ of the polynomial ring $\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[x^1, \cdots, x^n]$ is generated by n homogeneous polynomials $p^1, \cdots, p^n$ such that*
$$h = \deg p^1 > \deg p^2 \geq \cdots \geq \deg p^{n-1} > \deg p^n = 2.$$
2. *$\{\deg p^1, \ldots, \deg p^n\}$ does not depend on the choice of $p^1, \cdots, p^n$.*
3. *The eigenvalues of the Coxeter transformation c are*
$$\exp\left(2\pi\sqrt{-1}\,\frac{\deg p^1 - 1}{h}\right), \cdots, \exp\left(2\pi\sqrt{-1}\,\frac{\deg p^n - 1}{h}\right)$$

We obtain the Frobenius structrue $(\eta, \circ, e, E)$ in Theorem 4 as follows;

- (unit vector field e) $\quad e := \dfrac{\partial}{\partial p^1}$
- (Euler vector field E)

$$E := \sum_{i=1}^{n} \frac{1}{h} x^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} \frac{\deg p^i}{h} p^i \frac{\partial}{\partial p^i}$$

- (metric η) Let $g : \Omega^1_{\mathfrak{h}} \times \Omega^1_{\mathfrak{h}} \to \mathcal{O}_{\mathfrak{h}}$ be a non-degenerated $\mathcal{O}_{\mathfrak{h}}$-bilinear form induced by $I : L \times L \to \mathbb{Z}$ under a natural identification of $T^*_x\mathfrak{h} \cong \mathfrak{h}^*$, that is,

$$g(dx^i, dx^j) := I(\alpha_i, \alpha_j).$$

It induces a symmetric $\mathcal{O}_{\mathfrak{h}/W}$-bilinear form $g : \Omega^1_{\mathfrak{h}/W} \times \Omega^1_{\mathfrak{h}/W} \to \mathcal{O}_{\mathfrak{h}/W}$. Then, we define

$$\eta := \mathrm{Lie}_e g.$$

In order to define the product structure, we need the following

Theorem 6 (Saito–Yano–Sekiguchi).

Let $\nabla\!\!\!/$ be the Levi–Civita connection with respect to η. There exists $\nabla\!\!\!/$-flat W-invariant homogeneous polynomials $t^1, \cdots, t^n$ satisfying the conditions of Chevalley's Theorem.

- (product structure $\circ$) Let ∇ be the Levi-Civita connection with respect to $g : \Omega^1_{\mathfrak{h}/W} \times \Omega^1_{\mathfrak{h}/W} \to \mathcal{O}_{\mathfrak{h}/W}$. The product structure $\circ$ is defined by

$$C^k_{ij} := \frac{h}{\deg t^k - 1} \sum_{a=1}^{n} \eta\left(\frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^a}\right) \cdot g\left(dt^a, \nabla_{\frac{\partial}{\partial t^j}} dt^k\right)$$

for $i, j, k \in Q_0$, and

$$\frac{\partial}{\partial t^i} \circ \frac{\partial}{\partial t^j} := \sum_{k=1}^{n} C^k_{ij} \frac{\partial}{\partial t^k}.$$

Consider the ℓ–Kronecker quiver $Q = K_\ell$ with $\ell \geq 3$. Then c does not have finite order. What is h in the case of ℓ–Kronecker?

Let ρ be the spectral radius of the Coxeter transformation c:

$$\rho = \frac{\ell^2 - 2 + \sqrt{\ell^4 - 4\ell^2}}{2} \ (> 1).$$

Then, the diagonalization of c is given by

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} \exp\left(2\pi\sqrt{-1} \cdot \dfrac{\log \rho}{2\pi\sqrt{-1}}\right) & 0 \\ 0 & \exp\left(-2\pi\sqrt{-1} \cdot \dfrac{\log \rho}{2\pi\sqrt{-1}}\right) \end{pmatrix}.$$

Define

$$h := \frac{2\pi\sqrt{-1}}{\log \rho} \in \mathbb{C}\backslash\mathbb{R}.$$

Hence, the eigenvalues of c are given by

$$\exp\left(2\pi\sqrt{-1}\cdot\frac{h-1}{h}\right), \quad \exp\left(2\pi\sqrt{-1}\cdot\frac{1}{h}\right).$$

It is an analogue of the condition (3) in Theorem 5.

Based on this, we expect that there exist W-invariant homogeneous polynomials t^1 and t^2 satisfying

$$\text{“}\deg t^1 = h\text{''}, \quad \deg t^2 = 2.$$

In order to define the polynomial t^1 satisfying $\deg t^1 = h$, we consider a space X instead of the Cartan subalgebra $\mathfrak{h}$.

Define the set of imaginary roots Δ^{im} by

$$\Delta^{\mathrm{im}} := \{w(\alpha) \in L \mid w \in W,\ \alpha \in L \text{ s.t. } I(\alpha, \alpha_i) \leq 0\}.$$

Definition 7 (Ikeda).

Let $\mathcal{I}$ be the imaginary cone, namely, the closure of the convex hull of $\Delta_+^{\mathrm{im}} \cup \{0\}$.

Define an open subset $X \subset \mathfrak{h}$ by

$$X := \mathfrak{h} \setminus \bigcup_{\lambda \in \mathcal{I} \setminus 0} H_\lambda$$

and a regular subset $X^{\mathrm{reg}} \subset X$ by

$$X^{\mathrm{reg}} := X \setminus \bigcup_{\alpha \in \Delta_+^{\mathrm{re}}} H_\alpha,$$

where $H_\lambda := \{Z \in \mathfrak{h} \mid Z(\lambda) = 0\}$ is the orthogonal complex hyperplane of $\lambda \in \mathfrak{h}^* = L \otimes_{\mathbb{Z}} \mathbb{C}$.

The following theorem is one of the reasons why the space X is suitable:

Theorem 8 (Ikeda).

Let Q be an acyclic connected finite quiver and $\mathcal{D}_Q$ be the derived category of finite dimensional nilpotent $\Gamma_2 Q$-modules. Then there is a covering map

$$\mathrm{Stab}^\circ(\mathcal{D}_Q) \longrightarrow X^{\mathrm{reg}}/W,$$

where $\mathrm{Stab}^\circ(\mathcal{D}_Q)$ is a connected component of $\mathrm{Stab}(\mathcal{D}_Q)$.

In the ADE case, it is known that $\mathrm{Stab}^\circ(\mathcal{D}_{\vec{\Delta}}) \to \mathfrak{h}^{\mathrm{reg}}/W$ is the universal covering map.

In our case, X is given by

$$X = \mathbb{C}^2 \setminus \bigcup_{0 \le \lambda \le \infty} \{(x^1, x^2) \in \mathbb{C}^2 \mid x^1 = -\lambda x^2\}$$

after the change of coordinates along with the imaginary cone $\mathcal{I}$. The Weyl group $W = \langle r_1, r_2 \rangle$ action on the coordinate (x^1, x^2) of X (or $\mathfrak{h}$) is

$$\begin{aligned} r_1 \cdot (x^1, x^2) &= (\nu^{-1} x^2, \nu x^1) \\ r_2 \cdot (x^1, x^2) &= (\nu x^2, \nu^{-1} x^1), \end{aligned}$$

where $\nu = \sqrt{\rho}$.

Roughly speaking, W-invariant functions we expect are

$$\begin{cases} t^1 & = (x^1)^h - (x^2)^h \\ t^2 & = x^1 x^2, \end{cases}$$

where $x^h := \exp(h \log(x))$.

Note that t^1 is a multi-valued function on X.

Lemma 9.

The universal cover $\widetilde{X}$ of X is given by

$$\begin{array}{rccc} \widetilde{X} = & \{(y^1, y^2) \in \mathbb{C}^2 \mid \left|\operatorname{Im} y^1 - \operatorname{Im} y^2\right| < \pi\} & \longrightarrow & X \\ & (y^1, y^2) & \longmapsto & (e^{y^1}, e^{y^2}) \end{array}$$

Define the W-action on $\widetilde{X}$ by

$$r_1 \cdot (y^1, y^2) = (y^2 - \log \nu,\ y^1 + \log \nu)$$
$$r_2 \cdot (y^1, y^2) = (y^2 + \log \nu,\ y^1 - \log \nu),$$

then the covering map in the above Lemma is W-equivariant.

Definition 10.

Define a complex analytic space $\widetilde{X}/\!\!/W$ as follows:

- *The underlying space is the quotient space $\widetilde{X}/W$ and denote by $\pi : \widetilde{X} \to \widetilde{X}/W$ the quotient map .*
- *The structure sheaf is $\mathcal{O}_{\widetilde{X}/\!\!/W} := \pi_* \mathcal{O}_{\widetilde{X}}^W$, where $\mathcal{O}_{\widetilde{X}}^W$ is the W-invariant subsheaf of $\mathcal{O}_{\widetilde{X}}$.*

It is shown by Dimitrov–Katzarkov that $\mathrm{Stab}(\mathcal{D}^b(K_\ell)) \cong \mathbb{C} \times \mathbb{H}$ as complex manifolds, where $\mathbb{H} = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| > 0\}$.

Proposition 11.

$\widetilde{X}/\!\!/W$ has the structure of a complex manifold. Moreover, there exists an isomorphism

$$\widetilde{X}/\!\!/W \cong \mathrm{Stab}(\mathcal{D}^b(K_\ell))$$

as complex manifolds.

We expect that $\widetilde{X}/\!\!/W$ has a Frobenius structure in the points of view of Conjecture 1 and Theorem 8.

Main Theorem (Ikeda-O-Shiraishi-Takahashi).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ of rank 2 and dimension $1 - \frac{2}{h}$ on $\widetilde{X}/\!\!/W$ satisfying

1. *The intersection form g coincides with the generalized Cartan matrix A_{K_ℓ}.*
2. *The functions (t^1, t^2) defined by*

$$t^1 = e^{hy^1} - e^{hy^2}$$
$$t^2 = e^{y^1 + y^2}$$

   are W-invariant homogeneous and forms a flat coordinate system of the Frobenius structure.
3. *The Euler vector field E is given by*

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{2}{h} t^2 \frac{\partial}{\partial t^2}.$$

Main Theorem is an analogue of Theorem 4 in the case of the ℓ-Kronecker quiver.
This Frobenius structure $(\eta, \circ, e, E)$ is constructed in the same way of the case of finite Weyl group.
In particular, the Frobenius potential is given by

$$\mathcal{F} = -\frac{1}{2h}\left(t^1\right)^2 t^2 + \frac{1}{h^2-1}\left(t^2\right)^{h+1}.$$

The dimension $d = 1 - \dfrac{2}{h}$ of the Frobenius manifold is not real number.

Reference :

[1] T. Bridgeland, Y, Qiu and T. Sutherland, *Stability conditions and the A_2 quiver*, arXiv:1406.2566v4

[2] G. Dimitrov, L. Katzarkov, *Bridgeland stability conditions on wild Kronecker quivers.* Adv. Math. 352 (2019), 27–55.

[3] B. Dubrovin, *Geometry of 2d topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348.

[4] B. Dubrovin and Y. Zhang, *Extended Affine Weyl Groups and Frobenius Manifolds*, Compositio Math. 111 (1998) 167–219.

[5] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 247–318.

[6] A. Ikeda, *Stability conditions for preprojective algebras and root systems of Kac-Moody Lie algebras*, arXiv:1402.1392v1

[7] Y. Ishibashi, Y. Shiraishi, A. Takahashi *Primitive forms for affine cusp polynomials.* Tohoku Math. J. (2) 71 (2019), no. 3, 437–464.

[8] K. Saito, T. Yano and J. Sekiguchi, *On a certain generator system of the ring of invariants of a finite reflection group*, Comm. Algebra 8 (1980), no. 4, 373–408.

[9] K. Saito, *Period mapping associated to a primitive form*, Publ. RIMS, Kyoto Univ. 19 (1983) 1231–1264.

[10] K. Saito, *On a linear structure of the quotient variety by a finite reflexion group*, Publ. RIMS 1993 Volume 29 Issue 4 Pages 535–579.

[11] K. Saito, A. Takahashi, *From primitive forms to Frobenius manifolds.* From Hodge theory to integrability and TQFT tt*-geometry, 31–48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

[12] I. Satake, *Frobenius manifolds for elliptic root systems*, Osaka J. Math. 47 (2010) 301–330.

[13] I. Satake, A. Takahashi *Gromov-Witten invariants for mirror orbifolds of simple elliptic singularities.* Ann. Inst. Fourier (Grenoble) 61 (2011), no. 7, 2885–2907.

[14] Y. Shiraishi, A. Takahashi and K. Wada, *On Weyl Groups and Artin Groups Associated to Orbifold Projective Lines*, Journal of Algebra, 453 (2016), 249–290.

[15] A. Takahashi, *Matrix Factorizations and Representations of Quivers I*, arXiv:math/0506347

[16] A. Takahashi, *A Brief Introduction to Frobenius Manifolds*, Virtual Introductory Lectures on Frobenius Manifolds, March 30, 2021.

Thank you for your attention !